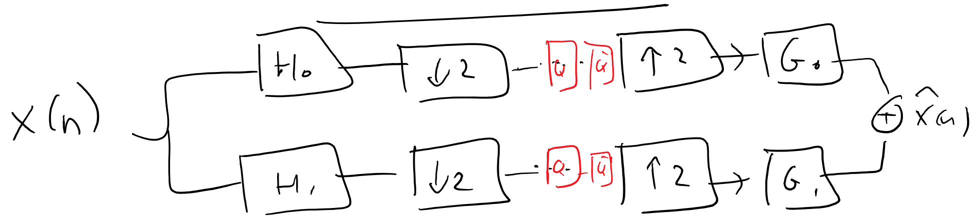
Subband coding

Monday, March 13, 2017 3:06 PM

Subband Coding



$$\hat{X}(\omega) = \frac{1}{2}\underbrace{\left[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right]}_{\text{Perfect reconstruction Term}} X(\omega)$$

$$+\ \frac{1}{2}\left[\underbrace{H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)}_{\text{aliasing term}}\right] X(\omega-\pi)$$

$$\hat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z)$$

$$+\ \frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z)$$

To eliminate aliasing $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$

or $H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega) = 0$

Proposed:

$$\left.\begin{aligned} G_0(\omega) &= H_1(\omega-\pi) \\ G_1(\omega) &= -H_0(\omega-\pi) \end{aligned}\right\}$$

$\Rightarrow$ Then aliasing is Removed

Problem: Assume

$$\left.\begin{aligned} H_0(\omega) &= H(\omega) \\ H_1(\omega) &= H(\omega-\pi) \end{aligned}\right\}$$

Then prototype filter H is enough to produce H_0, H_1, G_0, G_1

$$\left.\begin{aligned} H_0(\omega) &= H(\omega) \\ H_1(\omega) &= H(\omega-\pi) \\ G_0(\omega) &= H(\omega) \\ G_1(\omega) &= -H(\omega-\pi) \end{aligned}\right\} \rightarrow \text{Eqn } 1$$

$$G_0(\omega) = H(\omega)$$
$$G_1(\omega) = -H(\omega-\pi)$$

PR How do we design $H(\omega)$ so that PR is satisfied

Make $\frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] = z^{-K}$ delay

$\Rightarrow$ mean output is delayed version of input

Assuming Eqn 1 $\rightarrow H^2(\omega) - H^2(\omega-\pi) = 2e^{-j\omega K}$

$$H^2(z) - H^2(-z) = 2z^{-K}$$

for some K

Assume $H(\omega)$ is linear phase. delay $\frac{N-1}{2}$

$\Rightarrow H(\omega) = H_r(\omega)e^{-j\omega\frac{(N-1)}{2}}$ N = filter length

$$\frac{\hat{X}(\omega)}{X(\omega)} = \underbrace{\left[|H(\omega)|^2 - (-1)^{N-1}|H(\omega-\pi)|^2\right]}e^{-j\omega(N-1)}$$

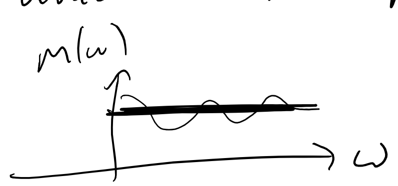
Define $M(\omega) = \left[|H(\omega)|^2 - (-1)^{N-1}|H(\omega-\pi)|^2\right]$

if N odd $\Rightarrow \left[M(\omega)\right]_{\omega=\pi/2} \Rightarrow M(\frac{\pi}{2}) = 0 \Rightarrow$ Bad

Ideally: $M(\omega) = 1 = |H(\omega)|^2 + |H(\omega-\pi)|^2$

only soln: $|H(\omega)|^2 = \cos^2 a\omega \rightarrow$ Trivial

Any nontrivial linear phase soln $\Rightarrow$ amplitude dist



Impossible have FIR, PR, linearphase

power complem filter

$$|H_0(\omega)|^2 + |H_1(\omega)|^2 = 1$$

Settle: Near Perfect Reconstruct

try $n(\omega)$ as flat as possible.

while minimize stop band ~~freq~~ energy

$$J = \alpha \int_{\omega_s}^{\pi} |H(\omega)|^2 \, d\omega + (1-\alpha) \int (n(\omega)-1)^2 \, d(\omega)$$

Optimize J w.r.t filter tap

Subject to $h(n)$, is symmetric impulse response $\Rightarrow$ linear phase

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April 6:

Consider 2 prototype filters

$$\rightarrow G_0(z) = H_1(-z) \qquad G_1(z) = -H_0(-z) \leftarrow$$

Define $\underline{P_0(z) = H_0(z) G_0(z)} \leftarrow$ PR

The Perfect Reconstruction condition

$$P_0(z) - P_0(-z) = 2 z^{-l} \leftarrow \qquad l = \text{delay integer}$$

or

$$z^{l} P_0(z) - z^{l} P_0(-z) = 2$$

Define $\underline{P(z) = z^{l} P_0(z)} \leftarrow$

- If H_0, H_1, G_0, G_1 are symmetric Then can show $P(z)$ is a symmetric polynomial

Then $P(z)$ can be written as:

$$P(z) = 1 + p_1(z + z^{-1}) + p_3(z^3 + z^{-3}) + p_5(z^5 + z^{-5}) + \cdots$$

- Compression want max # of zeros of $H_0(z)$ to be at $z = -1$ or $w = \pi$

— $z = -1$ is also a zero of $P_0(z)$ and $P(z)$

Let's write $P(z)$ as:

$$\boxed{P(z) = (1 + z^{-1})^m (1 + z)^m R(z)}$$

where $R(z)$ is a symmetric polynomial.

$$\Rightarrow R(z) = R(z^{-1})$$

Assume $R(z)$ is of this form:

$$\boxed{R(z) = r_0 + \sum_{s=1}^{m-1} r_s (z^s + z^{-s})}$$

Suppose $m = 1$ $\quad R(z) = r_0 = \frac{1}{2}$

$$\rightarrow P(z) = \frac{1}{2}(1+z)\underline{(1+z^{-1})} = \frac{1}{2}(z + 2 + z^{-1})$$
$$= z^{l} H_0(z) G_0(z)$$

What are some possibilities for H_0 and G_0

How about: $\quad \underline{H_0(z)} = \frac{1}{\sqrt{2}}(1 + z^{-1})$

$l = 1$

$$\Rightarrow P(z) = \frac{1}{2}(1+z)\underline{(1+z^{-1})} = z \frac{1}{\sqrt{2}}\underline{(1+z^{-1})}\, G_0(z)$$

$$G_0(z) = \frac{1}{\sqrt{2}}(1+z) z^{-1} = \frac{1}{\sqrt{2}}(z^{-1} + 1)$$

$\Longrightarrow$ Haar filters.

What if $m = 2$?

$$R(z) = r_0 + r_1(z + z^{-1}) = az + b + az^{-1}$$

$b = r_0 \quad r_1 = a$

$$\Rightarrow P(z) = (az + b + az^{-1})(1+z^{-1})^2(1+z)^2$$

$$= az^3 + (4a+b)z^2 + (7a+4b)z + (8a+6b) + (7a+4b)z^{-1} + (4a+b)z^{-2} + az^{-3}$$

$\Rightarrow$ Even powers of $P(z)$ must be zero (if symmetry)

$\Rightarrow$ Coeff of $z^0 = 1$ To be 1

$\Rightarrow \quad 4a+b=0 \qquad 8a+b = 1 \Rightarrow a = \frac{-1}{16} \quad b = 1/4$

$$P(z) = \frac{1}{16} z^3 (1+2z^{-1}+z^{-2})^2 (-1+4z^{-1}-z^{-2})$$

$$= z^l P_0(z) = z^l G_0(z) H_0(z)$$

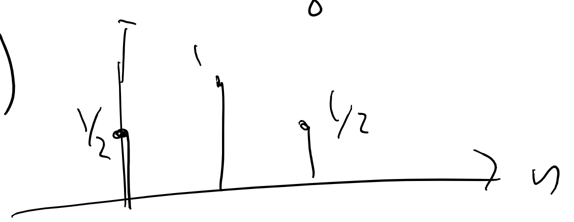
How about factoring $P(z)$. Assume $l=3$

(a) $H_0(z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2})$

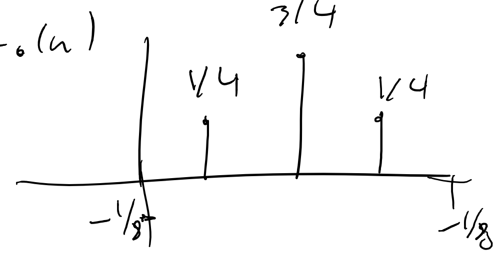
$$G_0(z) = \frac{1}{8}(1+2z^{-1}+z^{-2})(-1+4z^{-1}+z^{-2})$$

$$\rightarrow G_0(z) = \frac{1}{8}(-1+2z^{-1}+6z^{-2}+2z^{-3}-z^{-4})$$

$h_0(n)$

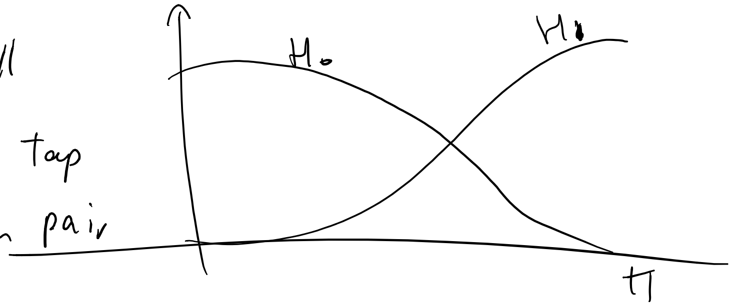


$G_0(n)$



$H_1(z) = G_0(-z) \qquad G_1(z) = -H_0(-z)$

Le Gall 3/5 tap filter pair



(b) Another factorization

$$H_0(z) = \frac{1}{8}\left(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4}\right)$$

$$G_0(z) = \frac{1}{2}\left(1 + 2z^{-1} + z^{-2}\right)$$

$$H_1(z) = \frac{1}{2}\left(1 - 2z^{-1} + z^{-2}\right)$$

$$G_1(z) = \frac{1}{8}\left(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}\right)$$

$H_0 \longrightarrow 5$ tap $\qquad G_0 \longrightarrow 3$ Tap

Le Gall 5/3 Tap filter

(c) another factorization $l = 3$

$$H_0(z) = \frac{1}{8}\left(1 + 3z^{-1} + 3z^{-2} + z^{-3}\right) \longleftarrow 4\text{ tap}$$

$$G_0(z) = \frac{1}{2}\left(-1 + 3z^{-1} + 3z^{-2} - z^{-3}\right) \leftarrow \text{4 taps}$$

$$H_1(z) = \frac{1}{2}\left(-1 - 3z^{-1} + 3z^{-2} + z^{-3}\right)$$

$$G_1(z) = \frac{1}{8}\left(-1 + 3z^{-1} - 3z^{-2} + z^{-3}\right)$$

Daubechie 4/4 Tap filter

MRA

Mallat

$\phi(x)$

$\psi(x)$

V, W

$V_0 \oplus W_0 \ldots$

Subband Coding

PR

linear phase

Signal Processing

$h\phi$

$h\psi$

Daubechies